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The Pinney equation and its discretization

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Abstract. The Pinney equation is part of the original Ermakov system which has been the subject of intensive study recently. Here we show that it may be related to a two-dimensional conformal Riccati equation leading to a new method for its linearization. A discrete analogue of the Pinney equation is constructed using the above connection with the conformal group. An alternative discretization is obtained by using a discrete Schwarz derivative. Both of these nonlinear difference equations are linearizable.

1. Introduction

Pinney (1950) stated without proof that the general solution of the second order nonlinear differential equation

$$v_{xx} + f(x)v + cv^{-3} = 0 \tag{1}$$

where c is a non-zero constant, is related to the fundamental set of solutions of the linear equation

$$\psi_{xx} + f(x)\psi = 0. \tag{2}$$

The equations (1) and (2) form a system of differential equations studied much earlier by Ermakov [1880]. Systems of this kind have been investigated intensively during recent years (Athorne 1992, Athorne *et al* 1990, Govinder and Leach 1994, Ray and Reid 1979) since they have applications in many areas including quantum mechanics, elasticity and optics. In fluid dynamics they arise in two-layer shallow water theory (Rogers 1989).

Recently Conte (1992) considered the generalized form

$$v_{xx} + g_1(x)v_x + g_2(x)v + h_3^2(x)v^{-3} = 0 \qquad h_3 \neq 0 \tag{3}$$

by making the substitution $u = v^{-2}$ to give

$$-\tfrac{1}{2}uu_{xx} + \tfrac{3}{4}u_x^2 - \tfrac{1}{2}g_1(x)uu_x + g_2(x)u^2 + h_3^2(x)u^4 = 0. \tag{4}$$

He showed that (4) has the Painlevé property if and only if

$$g_1(x) + h_{3,x}/h_3(x) = 0 \tag{5}$$

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and this is precisely the condition for (3) to transform to the Pinney equation (1) under the substitution

$$v(x) \to \mathrm{e}^{-\frac{1}{2}\int^x g_1(t)\,\mathrm{d}t} v(x) \equiv K^{-1} h_3^{1/2} v(x) \tag{6}$$

with $K^4 = c$:

$$f(x) \equiv g_2(x) + \frac{1}{2}\frac{h_{3,xx}}{h_3} - \frac{3}{4}\left(\frac{h_{3,x}}{h_3}\right)^2. \tag{7}$$

In that case, on making the substitution

$$u(x) = \phi_x/[2h_3(x)\phi(x)] \tag{8}$$

equation (4) transforms to

$$\frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 = 2f(x). \tag{9}$$

This is a particularly interesting form of the Pinney equation as the Schwarz derivative appears on the l.h.s., leading to a linearization as described in section 2. In section 3 we establish a very useful association between the Pinney equation and a two-dimensional conformal Riccati system which lead to further methods for linearizing (3) when (5) is satisfied.

At the present time there is great interest in discretizing differential equations of Painlevé type. The Pinney equation has this property and belongs to the class of Gambier (1910) equations which are linearizable. In section 4 we obtain discrete versions of the Pinney equation which retain this property of linearizability. The first uses the fact that the conformal Riccati equation corresponding to (4) is linked with an infinitesimal conformal transformation as discussed by Anderson *et al* (1982). The obvious way to proceed is to take instead a discrete conformal transformation and corresponding difference analogues of the conformal Riccati equations. A discrete form of the generalized Pinney equation (4) may then be constructed from these nonlinear difference equations.

A second method of discretization for (4) is to start from its Schwarzian form (7). There are many possible discrete forms of the Schwarzian derivative and we will introduce one that is related to a discrete Riccati equation and corresponding second order linear difference equation in analogy to the continuous case.

In section 5 we will summarize and give conclusions to our work.

2. The Pinney equation and three fundamental ODE

Let S be a given analytic function of the complex variable x and consider the three differential equations in ϕ, ω, ψ, respectively:

$$\frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 = S(x) \tag{10}$$

$$\omega_x + \omega^2 = -S(x)/2 \tag{11}$$

$$-2\psi_{xx}/\psi = S(x). \tag{12}$$

The expression on the l.h.s. of (10) is the Schwarz derivative (see Hille 1976, ch 10) of ϕ. It is important since it is the unique elementary function of the derivatives which is invariant under homographic transformation of ϕ:

$$\phi \to \frac{a\phi + b}{c\phi + d} \qquad (a, b, c, d) \text{ arbitrary constants } ad - bc = 1$$

and hence plays a central role in 'invariant' Painlevé analysis (Conte 1989).

It is a classical result (Painlevé 1897) that these three equations are equivalent. There are six relations between ϕ, ω, ψ (Conte 1992) such that if ψ is a solution of (12) then ϕ, ω are corresponding solutions of (10), (11), respectively, and similarly for ϕ and ω. They take the form

$$\omega(\phi) = \frac{c_1\phi_x}{c_1\phi + c_2} - \frac{\phi_{xx}}{2\phi_x} \tag{13}$$

$$\psi(\phi) = (c_1\phi + c_2)\phi_x^{-\frac{1}{2}} \tag{14}$$

$$\phi(\omega) = \frac{c_1(\omega_2 - \omega_1) + c_2(\omega_3 - \omega_1)}{c_3(\omega_2 - \omega_1) + c_4(\omega_3 - \omega_1)} \qquad c_1c_4 - c_2c_3 = 1 \tag{15}$$

$$\psi(\omega) = c_1\psi_1 + c_2\psi_2 \qquad \psi_1^2 = \frac{(\omega_2 - \omega_3)}{(\omega_2 - \omega_1)(\omega_3 - \omega_1)} \qquad \psi_2 = \frac{\psi_1(\omega_3 - \omega_1)}{(\omega_3 - \omega_2)} \tag{16}$$

$$\phi(\psi) = \frac{c_1\psi_1 + c_2\psi_2}{c_3\psi_1 + c_4\psi_2} \qquad c_1c_4 - c_2c_3 = 1 \tag{17}$$

$$\omega(\psi) = \frac{c_1\psi_{1,x} + c_2\psi_{2,x}}{c_1\psi_1 + c_2\psi_2} \tag{18}$$

where $\omega_1, \omega_2, \omega_3$ and ψ_1, ψ_2 are respectively particular solutions of (11) and (12).

We have not been able to obtain discrete forms of all these relations, but in section 4 we will find enough to take us from a discrete version of (10) to a corresponding version of (12) and back again. This then provides a linearization of the discrete form we have constructed for (10) and hence for the generalized Pinney equation (4).

3. Conformal Riccati system

Two main classes of linearizable coupled Riccati systems of equations are (Anderson *et al* 1982) as follows:

(1) Projective Riccati equations:

They have the matrix form

$$\boldsymbol{\omega}_x = \boldsymbol{a} + B\boldsymbol{\omega} + \boldsymbol{\omega}(\boldsymbol{c}, \boldsymbol{\omega}) \tag{19}$$

where the elements of $N \times N$ matrix B and N-dimensional vectors $\boldsymbol{a}, \boldsymbol{c}$ are functions of the independent variable x.

(2) Conformal Riccati equations:

$$\boldsymbol{\omega}_x = \boldsymbol{\beta} + E\boldsymbol{\omega} + a\boldsymbol{\omega} + \boldsymbol{\omega}(\boldsymbol{\gamma}, \boldsymbol{\omega}) - \tfrac{1}{2}\boldsymbol{\gamma}(\boldsymbol{\omega}, \boldsymbol{\omega}) \tag{20}$$

with $N \times N$ matrix E satisfying

$$E\tilde{I} + \tilde{I}E^T = 0 \tag{21}$$

where $\tilde{I}$ is such that

$$(\boldsymbol{\delta}, \boldsymbol{\alpha}) \equiv \boldsymbol{\delta}^T\tilde{I}\boldsymbol{\alpha} = \delta_1\alpha_1 + \delta_2\alpha_2 + \cdots + \delta_p\alpha_p - \delta_{p+1}\alpha_{p+1} - \cdots - \delta_N\alpha_N \tag{22}$$

and $1 \leqslant p \leqslant N$.

We will now show that the Pinney equation in the form (4) corresponds to a two-dimensional conformal Riccati system. Take $N = 2$, $\tilde{I} \equiv \begin{bmatrix} 1 & 0 \\ 0 & \epsilon^2 \end{bmatrix}$ and with $\epsilon^2 = \pm 1$. Then (20) may be written in component form,

$$\begin{bmatrix} y_x \\ z_x \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + \begin{bmatrix} B_1 & B_2 \\ -\epsilon^2 B_2 & B_1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} y \\ z \end{bmatrix}[D_1 D_2]\begin{bmatrix} y \\ \epsilon^2 z \end{bmatrix} - \frac{1}{2}\begin{bmatrix} D_1 \\ D_2 \end{bmatrix}[yz]\begin{bmatrix} y \\ \epsilon^2 z \end{bmatrix} \tag{23}$$

where all coefficients are real functions of x. Choosing $D_1 = B_1 = B_2 \equiv 0$ and eliminating z from the two coupled first order ODEs corresponding to (23), we obtain

$$yy_{xx} - \frac{3}{2}y_x^2 + \left[2A_1 - \frac{D_{2,x}}{D_2}y\right]y_x - \frac{1}{2}A_1^2 + y\left[-A_{1,x} + A_1\frac{D_{2,x}}{D_2}\right]$$
$$-y^2[\epsilon^2 A_2 D_2] + \frac{\epsilon^2}{2}D_2^2 y^4 = 0. \tag{24}$$

We see that this is equivalent to (4) when

$$A_1 \equiv 0 \qquad A_2 = 2g_2\epsilon^2/D_2 \qquad D_2^2 = -4h_3^2\epsilon^2 \tag{25}$$

and that (5) is then also satisfied. As D_2 is real, $\epsilon^2 h_3^2$ must be negative.

The standard way to linearize the conformal Riccati system (20) is to first of all convert it to a projective Riccati system of one higher dimension. For the two-dimensional case (23) we make the definition

$$w \equiv y^2 + \epsilon^2 z^2 \tag{26}$$

and then in the case where the coefficients satisfy (25),

$$y_x = \epsilon^2 D_2 yz \tag{27}$$
$$z_x = A_2 - \tfrac{1}{2}D_2 w + \epsilon^2 D_2 z^2 \tag{28}$$
$$w_x = 2\epsilon^2 A_2 z + \epsilon^2 D_2 wz \tag{29}$$

which is a three-dimensional projective Riccati system. This is equivalent to the linear system

$$\begin{bmatrix} v_{1,x} \\ v_{2,x} \\ v_{3,x} \\ v_{4,x} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}D_2 & A_2 \\ 0 & 2\epsilon^2 A_2 & 0 & 0 \\ 0 & -\epsilon^2 D_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \tag{30}$$

when $y = v_1/v_4$, $z = v_2/v_4$, $w = v_3/v_4$.

A very elegant way of linearizing the two-dimensional conformal Riccati equation (23) is to note that it corresponds to a scalar Riccati equation. If we define $W \equiv z + \mathrm{i}\epsilon y$, then (23) with the coefficients satisfying (25), correspond to

$$W_x = A_2 + \frac{\epsilon^2}{2}D_2 W^2. \tag{31}$$

This scalar equation may be linearized in the usual way by setting $W = \frac{-\epsilon^2}{D_2}\{2\frac{\psi_x}{\psi} + \frac{D_{2,x}}{D_2}\}$ and using relations (25) to obtain

$$\psi_{xx} + \frac{S}{2}\psi = 0 \tag{32}$$

where $S(x) \equiv 2f(x)$ is the r.h.s. of the Schwarzian form (7), (9) of the Pinney equation. Let g_2, $h_3^2 \in \mathbb{R}$ so that S is real, and take $\psi = r\mathrm{e}^{\mathrm{i}\epsilon\eta}$ with r, η real. Then (32) becomes

$$r_{xx} + \left(\frac{S}{2} - \epsilon^2\eta_x^2\right)r + \mathrm{i}\epsilon(2r_x\eta_x + r\eta_{xx}) = 0 \tag{33}$$

and will be satisfied if

$$r_{xx} + \left(\frac{S}{2} - \epsilon^2\eta_x^2\right)r = 0 \tag{34}$$

and

$$\frac{\eta_{xx}}{\eta_x} = -2\frac{r_x}{r} \Rightarrow \eta_x = \epsilon\sqrt{-C}r^{-2} \tag{35}$$

where C is an arbitrary constant, and hence when

$$r_{xx} + \frac{S}{2} r + C r^{-3} = 0. \tag{36}$$

Here appears the link with the original Pinney equation (1).

Setting $\psi = \psi_1 + \mathrm{i}\epsilon\psi_2$ where ψ_j are two linearly independent real solutions of (32) with Wronskian equal to $\epsilon\sqrt{-C}$, we have that $r^2 = \psi_1^2 + \epsilon^2\psi_2^2$ and we directly derive the solution of (4) by taking account of (6):

$$u = \frac{\sqrt{C}}{r^2 h_3}. \tag{37}$$

As the Wronskian is invariant under the linear transformation

$$\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \to \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \tag{38}$$

with $(ad - bc)^2 = 1$, the general solution of (4) is

$$u = \frac{\sqrt{C}}{h_3(A_1\psi_1^2 + 2A_2\psi_1\psi_2 + A_3\psi_2^2)} \tag{39}$$

with $A_1 A_3 - A_2^2 = \epsilon^2(ad - bc)^2 = \epsilon^2$. This form coincides with the formula given by Athorne (1992).

4. Discrete forms of the Pinney equation

There are of course infinitely many ways of constructing nonlinear difference equations which tend to the Pinney equation in the continuum. We will discuss two approaches based on the connection with the conformal Riccati equation and Schwarz derivative, respectively.

The conformal Riccati equations in (20) are the infinitesimal counterpart (Anderson *et al* 1982) of the discrete conformal transform of the vector $\boldsymbol{\omega}$ in $\mathbb{R}^N$ given by

$$\boldsymbol{\omega} \to \mathrm{e}^{\rho}\Lambda \ \frac{[\boldsymbol{\omega} + \boldsymbol{\gamma}\boldsymbol{\omega}^2]}{[1 + 2(\boldsymbol{\omega}, \boldsymbol{\gamma}) + \boldsymbol{\omega}^2\boldsymbol{\gamma}^2]} + \boldsymbol{\alpha} \tag{40}$$

where Λ is a general Lorentz transformation and $\boldsymbol{\gamma}, \boldsymbol{\alpha}$ are in $\mathbb{R}^N$ and ρ is a scalar. The most obvious and natural way to discretize (20) is then to consider (40) as a mapping from $\boldsymbol{\omega}(n)$ to $\boldsymbol{\omega}(n+1)$. To obtain the discrete analogy of the Pinney equation we take $N = 2$ as in section 3 and the standard form of (40) with $\Lambda = I$, $\rho = 0$, $\boldsymbol{\gamma} = \binom{0}{\gamma_2}$, $\boldsymbol{\alpha} = \binom{0}{\alpha_2}$. The mapping (40) may then be written in the component form

$$y(n+1) = y(n)/\{1 + 2\epsilon^2\gamma_2(n)z(n) + [y^2(n) + \epsilon^2 z^2(n)]\epsilon^2\gamma_2^2(n)\} \tag{41}$$

$$\begin{aligned} z(n+1) = {} & \{z(n) + \gamma_2(n)[y^2(n) + \epsilon^2 z^2(n)]\} \\ & \times\{1 + 2\epsilon^2\gamma_2(n)z(n) + [y^2(n) + \epsilon^2 z^2(n)]\epsilon^2\gamma_2^2(n)\}^{-1} + \alpha_2(n) \end{aligned} \tag{42}$$

and one obtains (23) in standard form (27), (28) by setting

$$\begin{aligned} & x = nh,\ y(n) \to y(x) \qquad z(n) \to z(x) \\ & \gamma_2(n) = -hD_2(x)/2 \qquad \alpha_2(n) = hA_2(x) \end{aligned} \tag{43}$$

and taking the continuum limit $h \to 0$.

By choosing homogeneous coordinates on the null cone in $\mathbb{R}^{N+2}$:

$$\omega_i = \xi_i/[\xi_0 + \xi_{N+1}] \qquad i = 1, \ldots, N \tag{44}$$

with

$$\sum_{j=0}^{p} \xi_j^2 - \sum_{j=p+1}^{N+1} \xi_j^2 = 0 \tag{45}$$

the mapping (40) corresponds to the linear transformation

$$\begin{pmatrix} \xi_0 \\ \boldsymbol{\xi} \\ \xi_{N+1} \end{pmatrix} \to g(\rho, \Lambda, \boldsymbol{\alpha}, \boldsymbol{\gamma}) \begin{pmatrix} \xi_0 \\ \boldsymbol{\xi} \\ \xi_{N+1} \end{pmatrix} \tag{46}$$

where

$$g(\rho, \Lambda, \boldsymbol{\alpha}, \boldsymbol{\gamma}) = g_T(\boldsymbol{\alpha}) g_D(\rho) g_L(\Lambda) g_C(\boldsymbol{\gamma}) \tag{47}$$

with $g_T(\boldsymbol{\alpha})$, $g_D(\rho)$, $g_L(\Lambda)$ and $g_C(\boldsymbol{\gamma})$ representing translations, dilations, Lorentz transformations and special conformal transformations respectively (Anderson *et al* 1982).

For the special case $N = 2$, $\Lambda = I$, $\rho = 0$, $\boldsymbol{\gamma} = \binom{0}{\gamma_2}$ and $\boldsymbol{\alpha} = \binom{0}{\alpha_2}$ considered here, g_D and g_L are unit matrices and

$$\begin{aligned} g_T(\boldsymbol{\alpha}) &= \begin{bmatrix} 1 + \frac{\epsilon^2}{2}\alpha_2^2 & 0 & \epsilon^2\alpha_2 & \frac{\epsilon^2}{2}\alpha_2^2 \\ 0 & 1 & 0 & 0 \\ \alpha_2 & 0 & 1 & \alpha_2 \\ -\frac{\epsilon^2}{2}\alpha_2^2 & 0 & -\epsilon^2\alpha_2 & 1 - \frac{\epsilon^2}{2}\alpha_2^2 \end{bmatrix} \\ g_C(\boldsymbol{\gamma}) &= \begin{bmatrix} 1 - \frac{\epsilon^2}{2}\gamma_2^2 & 0 & \epsilon^2\gamma_2 & \frac{\epsilon^2}{2}\gamma_2^2 \\ 0 & 1 & 0 & 0 \\ -\gamma_2 & 0 & 1 & \gamma_2 \\ -\frac{\epsilon^2}{2}\gamma_2^2 & 0 & \epsilon^2\gamma_2 & 1 + \frac{\epsilon^2}{2}\gamma_2^2 \end{bmatrix}. \end{aligned} \tag{48}$$

Then (41), (42) are equivalent to the linear recurrence relations

$$\begin{pmatrix} \xi_0(n+1) \\ \xi_1(n+1) \\ \xi_2(n+1) \\ \xi_3(n+1) \end{pmatrix} = g_T(\boldsymbol{\alpha}) g_C(\boldsymbol{\gamma}) \begin{pmatrix} \xi_0(n) \\ \xi_1(n) \\ \xi_2(n) \\ \xi_3(n) \end{pmatrix} \tag{49}$$

with

$$y(n) = \xi_1(n)/[\xi_0(n) + \xi_3(n)] \qquad z(n) = \xi_2(n)/[\xi_0(n) + \xi_3(n)]. \tag{50}$$

A discrete form of the Pinney equation may then be obtained by eliminating $z(n)$ between (41), (42). From the first of these we have a quadratic equation for $z(n)$ and we choose the root

$$z(n) = \frac{\epsilon^2 \left(-1 + \sqrt{\frac{y(n)}{y(n+1)} - \epsilon^2\gamma_2^2(n) y^2(n)}\right)}{\gamma_2(n)} \tag{51}$$

to get the correct continuum limit. Substituting in (42), we obtain the equation

$$\begin{aligned} \frac{1}{\gamma_2(n+1)} + \frac{1}{\gamma_2(n)} + \epsilon^2\alpha_2(n) &= \frac{1}{\gamma_2(n+1)} \sqrt{\frac{y(n+1)}{y(n+2)} - \epsilon^2\gamma_2^2(n+1) y^2(n+1)} \\ &\quad + \frac{y(n+1)}{y(n)\gamma_2(n)} \sqrt{\frac{y(n)}{y(n+1)} - \epsilon^2\gamma_2^2(n) y^2(n)}. \end{aligned} \tag{52}$$

The square roots may be eliminated by squaring twice but this leads to rather complicated expressions. The nonlinear difference equation (52) can be considered as the discrete analogue of the Pinney equation. It is simplest to consider coupled recurrence relations (41), (42) which are linearizable and also have the inverse transformation:

$$\begin{aligned} y(n) = y(n+1)\{1 - 2\epsilon^2\gamma_2(n)[z(n+1) - \alpha_2(n)] \\ +\epsilon^2[y^2(n+1) + \epsilon^2(z(n+1) - \alpha_2(n))^2]\gamma_2^2(n)\}^{-1} \end{aligned} \tag{53}$$

$$\begin{aligned} z(n) = \{z(n+1) - \alpha_2(n) - \gamma_2(n)[y^2(n+1) + \epsilon^2(z(n+1) - \alpha_2(n))^2]\} \\ \times\{1 - 2\epsilon^2\gamma_2(n)[z(n+1) - \alpha_2(n)] \\ +\epsilon^2[y^2(n+1) + \epsilon^2(z(n+1) - \alpha_2(n))^2]\gamma_2^2(n)\}^{-1}. \end{aligned} \tag{54}$$

The nonlinear difference equation (52) is a discrete form of the Pinney equation which it tends to in the contiuum limit. This is most easily seen by setting $y(n) = 1/w(n)$ and taking the limiting process (43). Expanding $w(n+2)$, $w(n+1)$ up to terms of $\mathrm{O}(h^2)$, it is a lengthy but straightforward calculation to show that in the limit $h \to 0$,

$$ww_{xx} - \frac{1}{2}w_x^2 + 2h_3^2 - ww_x\frac{h_3'(x)}{h_3(x)} + 2g_2(x)w^2 = 0 \tag{55}$$

which corresponds to (3) and (5) in the variable $w = v^2$.

An alternative way of discretizing the Pinney equation is to start from its form (9) in terms of the Schwarzian derivative of ϕ. Although there are a number of discrete forms for this derivative with the property of invariance under the homographic transformation

$$\phi(n) \to [a\phi(n) + b]/[c\phi(n) + d] \tag{56}$$

we will choose a particular form where the discrete analogue of (10) may be transformed into the discrete Riccati equation corresponding to (11).

The Schwarzian derivative of the function $\phi(n)$ of the discrete variable n is defined by

$$\begin{aligned} S(n) \equiv 4\{[\phi(n) - \phi(n-1)][\phi(n+1) - 3\phi(n) + 3\phi(n-1) - \phi(n-2)] \\ -\tfrac{3}{2}[\phi(n+1) - 2\phi(n) + \phi(n-1)][\phi(n) - 2\phi(n-1) + \phi(n-2)]\} \\ \times\{[\phi(n+1) - \phi(n-1)][\phi(n) - \phi(n-2)]\}^{-1} \end{aligned} \tag{57}$$

and corresponds to that given previously by Faddeev and Takhtajan (1986).

It is easily seen that this has the correct continuum limit (10) and very interestingly is related to the cross-ratio of four adjacent values of ϕ, i.e.

$$S(n) = -2\left\{1 - 4\frac{[\phi(n-1) - \phi(n-2)][\phi(n+1) - \phi(n)]}{[\phi(n+1) - \phi(n-1)][\phi(n) - \phi(n-2)]}\right\} \tag{58}$$

so that $S(n)$ is obviously invariant under the homographic transformation of $\phi(n)$.

In analogy to the continuum case, (57) is equivalent to the discrete Riccati equation

$$\omega(n+1) - \omega(n) + \omega(n)\omega(n+1) = -\tfrac{1}{2}S(n) \tag{59}$$

where

$$\omega(n) \equiv -[\phi(n) - 2\phi(n-1) + \phi(n-2)]/[\phi(n) - \phi(n-2)] \tag{60}$$

corresponding to definition of $\omega(x)$ in (13) with $c_1 = 0$. It should be remarked that the denominator on the r.h.s. of (13) is replaced by a sum of two first differences of $\phi(n)$ rather than twice a single first difference.

In the continuous case, the generalized Pinney equation (4) with coefficients satisfying (5) may be written in the form (9), where the l.h.s. is the Schwarzian of $\phi(x)$ after substituting u by $\phi_x/2h_3\phi$. The discrete analogue of (7), (9) is then from (57),

$$\begin{aligned}
4\{&[\phi(n)-\phi(n-1)][\phi(n+1)-3\phi(n)+3\phi(n-1)-\phi(n-2)]\\
&-\tfrac{3}{2}[\phi(n+1)-2\phi(n)+\phi(n-1)][\phi(n)-2\phi(n-1)+\phi(n-2)]\}\\
&\times\{[\phi(n+1)-\phi(n-1)][\phi(n)-\phi(n-2)]\}^{-1}\\
&=2g_2(n)+\frac{[h_3(n+1)-2h_3(n)+h_3(n-1)]}{h_3(n)}\\
&\quad-\frac{3[h_3(n+1)-h_3(n)][h_3(n)-h_3(n-1)]}{2h_3(n-1)h_3(n+1)}. \qquad (61)
\end{aligned}$$

In analogy to the continuous case, we make the substitution

$$u(n)=[\phi(n)-\phi(n-1)]/2h_3(n)\phi(n) \qquad (62)$$

in (61) and using (58) to give

$$\begin{aligned}
\{&16h_3(n-1)h_3(n+1)u(n+1)u(n-1)[1-2h_3(n)u(n)]\}\\
&\times\{[1-(1-2h_3(n)u(n))(1-2h_3(n+1)u(n+1))]\\
&\times[1-(1-2h_3(n-1)u(n-1))(1-2h_3(n)u(n))]\}^{-1}-1\\
&=g_2(n)+\frac{[h_3(n+1)-2h_3(n)+h_3(n-1)]}{h_3(n)}\\
&\quad-\frac{3[h_3(n+1)-h_3(n)][h_3(n)-h_3(n-1)]}{4h_3(n-1)h_3(n+1)}. \qquad (63)
\end{aligned}$$

We can show that (63) has the correct continuum limit in the following manner. Set $u(n)=h/[w(n)+hh_3(n)]$ and then (63) is transformed to

$$\frac{w^2(n)}{h_3^2(n)}-h^2-(1+h^2f(n))\left[\frac{w(n)}{2h_3(n)}+\frac{w(n+1)}{2h_3(n+1)}\right]\left[\frac{w(n)}{2h_3(n)}+\frac{w(n-1)}{2h_3(n-1)}\right]=0 \qquad (64)$$

and $h^2f(n)$ is the r.h.s. of (63).

Setting

$$x=nh \qquad w(n)\to w(x) \qquad f(n)\to f(x) \qquad h_3(n)\to h_3(x)$$

and expanding $w(x\pm h)$, $h_3(x\pm h)$ to $O(h^2)$, we obtain the continuous limit (55).

The equation (63) is our second type of discrete Pinney equation. It has the advantage over the previous form (52) in that no square roots have to be eliminated and it may be linearized since it is equivalent to the discrete Riccati equation (59) where $S(n)$ is the r.h.s. of (61). This in turn may be transformed to a second order linear difference equation through the substitution

$$\omega(n)=[\psi(n+1)-\psi(n)]/\psi(n) \qquad (65)$$

giving

$$\psi(n+2)-2\psi(n+1)+\psi(n)=-\tfrac{1}{2}S(n)\psi(n). \qquad (66)$$

Given a solution of (59), we have from (60) that

$$\omega(n)=-\frac{[\Delta(n)-\Delta(n-1)]}{[\Delta(n)+\Delta(n-1)]} \qquad \Delta(n)\equiv\phi(n)-\phi(n-1). \qquad (67)$$

The solution of this equation for $\Delta(n)$ is

$$\Delta(n) = \prod_{j=1}^{n} \left[\frac{1-\omega(j)}{1+\omega(j)} \right] \Delta(0) \qquad n = 1, 2, \ldots \tag{68}$$

$$= \prod_{j=n}^{-1} \left[\frac{1+\omega(j+1)}{1-\omega(j+1)} \right] \Delta(0) \qquad n = -1, -2, \ldots \tag{69}$$

and then

$$\phi(n) = \sum_{j=1}^{n} \Delta(j) + \phi(0) \qquad n = 1, 2, \ldots \tag{70}$$

$$= -\sum_{j=n}^{-1} \Delta(j+1) + \phi(0) \qquad n = -1, -2, \ldots . \tag{71}$$

Finally the general solution of our second discrete Pinney equation (63) is

$$u(n) = \frac{\Delta(n)}{2h_3(n)\phi(n)} \qquad n = 0, \pm 1, \pm 2, \ldots \tag{72}$$

where $\Delta(n)$, $\phi(n)$ are given by (68) to (70) in terms of the general solution $\omega(n)$ of the discrete Riccati equation (59).

We finally investigate the continuum limit of (68) by writing it in the form

$$\frac{1}{h}[\phi(nh) - \phi((n-1)h)] = \frac{1}{h}[\phi(0) - \phi(-h)] \prod_{j=1}^{n} \left[\frac{1-h\omega(jh)}{1+h\omega(jh)} \right]$$

$$= \frac{1}{h}[\phi(0) - \phi(-h)] \exp\left\{ \sum_{j=1}^{n} \log\left[\frac{1-h\omega(jh)}{1+h\omega(jh)} \right] \right\} \tag{73}$$

with $h = 1$. Setting $x = nh$, $t = jh$ and letting $h \to 0$, we obtain

$$\phi_x(x) = \phi_x(0) \exp\left[-2\int_0^x \omega(t)\,\mathrm{d}t \right]. \tag{74}$$

Taking logarithms and then differentiating, we obtain precisely the expression for $\omega(x)$ in terms of the derivatives of $\phi(x)$ given by (13) with $c_1 = 0$.

5. Conclusions

The classes of N first order coupled nonlinear differential equations with superposition principles have been identified for $N \leqslant 3$ by Bountis *et al* (1986). For $N = 2$ there are just two classes which are, respectively, projective Riccati and conformal Riccati systems. We have shown in section 3 that the Pinney equation is equivalent to such a conformal Riccati system, leading to a novel linearization of this equation. This connection has also been used in section 4 to obtain a discrete Pinney equation from a corresponding discrete conformal Riccati system given by (41), (42) or equivalently the difference equation (52). This approach has the advantage that this discrete Pinney equation corresponds to the mapping (41), (42) which is linearizable and invertible. We are now looking at the possibility of linking higher-dimensional conformal Riccati systems to coupled Pinney systems such as those being studied by Athorne (1992).

Our alternative approach to obtain a discrete analogue of the Pinney equation is to work through the expression in terms of a Schwarzian derivative. We defined a discrete version of this derivative, which interestingly is related to the cross derivative of four adjacent values of the dependent variable. The corresponding nonlinear difference equation was shown to be equivalent to a discrete Riccati system and hence linearizable. This approach has the advantage that the resulting discrete Pinney equation (63) contains no square roots to be eliminated, as was the case with (52).

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